SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

SOLUTIONS of problems in No. 4 have been received as follows:

From R. J. Adcock, 315; David Barrow, Jr., 319; Prof. W. P. Casey, 313, 319; George Eastwood, 313, 314, 316, 319; William Hoover, 317; G. H. Harvill, 318; Chas. H. Kummell, 315; Prof. E. B. Seitz, 317, 318; Prof. J. Scheffer, 317, 319.

313. "A piston, weight w, is dropped into the end of a vertical cylinder filled with air, length l; how far will the piston descend, assuming no friction nor escape of air, nor heat from the compressed air?"

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Let r = radius of cylinder, l = its length, w = weight of the piston, ε = elasticity of *any* fluid, with which the cylinder may be filled, and filling a given length h. Then, for any varyable length x, of the cylinder, the elasticity for the space $\pi r^2 x$ is $\varepsilon \times (h \div x)$; the moving force upon the piston is $w - \varepsilon \times (h \div x)$ and the accelerating force is

$$F = 1 - \frac{\varepsilon}{w} \cdot \frac{h}{x}.$$

Now from the formula for accelerating forces we have

$$\int v dv = -\int F dx,$$

from which we deduce

$$v^2 = 2\left[\frac{\varepsilon h}{w} . \log x - x\right] + C = \left[\frac{\varepsilon h}{w} . \log \frac{x}{h} + h - x\right].$$

Let s = space through which the piston descends to acquire the velocity v, and let $x = h - s$. Then

$$v^2 = 2\left[\frac{\varepsilon h}{w} . \log \frac{h-s}{h} + s\right].$$

Suppose now the fluid to be atmosphere with an elastic pressure of 15lbs upon the square inch; we shall have in this case,

$$\varepsilon = \pi r^2 \times 15 \text{ lbs}.$$

Adding to w the atmospheric pressure upon it, we get

$$v^2 = 2\left[\frac{15\pi r^2 h}{15\pi r^2 + w} . \log \frac{h-s}{h} + s\right],$$

from which to determine s. In this case we may write l for h; and because when the accelerating force becomes zero the descent of the piston ceases,

$$F = 1 - \frac{15\pi r^2 l}{(15\pi r^2 + w)x} = 0. \quad \text{Hence } x = \frac{15\pi r^2 l}{15\pi r^2 + w}.$$

314. "A helix is coiled about a right cylinder the radius of whose base is 1. The projection of the evolute of the helix on the plane of the base of the cylinder encloses double the area of that base. Required the angle made with the plane of the base by the tangent to the helix."

SOLUTION BY EASTWOOD.

The evolute of the helix is the contour of the osculating surface of the helix. If the projection of the contour of this surface be required on the plane of $x\,y$, it will be found to be the involute of the projection of the helix on the base of the cylinder. (De Morgan's Calculus, p. 416.) For a general solution, let a = radius of the cylinder, h = step of the helix, r, θ, the coordinates of any point of the involute, and p the perpendicular on the tangent. We have for the polar equation

$$a\theta = \sqrt{(r^2-a^2)} - a\cos^{-1}(a\div r)\,;$$

and for the area

$$A = \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int\frac{prdr}{(r^2-p^2)^{\frac{1}{2}}} = \frac{1}{2a}\int(r^2-a^2)^{\frac{1}{2}}rdr,$$
$$= \tfrac{4}{3}\pi^3a^2,$$

where p is taken between the limits $p = 0$, and $p = 2\pi a$.

When $a = 1$, this result gives $\frac{4}{3}\pi^3$ for the area enclosed by the projection of the evolute of the helix on the base of the cylinder, which differs from that specified in the enunciation of the question.

The angle which the tangent to the helix makes with the plane of xy is found from the eq'n $\tan\varphi = h\div 2\pi a$, whatever value may be given to a.

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315. "Two marksmen compete for a prize. Three shots are to be fired by each. The distance of each shot hole from the bull's eye is measured and he is the winner whose total distance is least. A's three shots measured respectively 1, 2, 3in. from the bull's eye. B then made three shots and the bull's eye fell off leaving no mark by which it could be replaced.

An examination showed that the coordinates of B's shot holes, referred to an origin arbitrarily taken, were (0, 0) in. (1, 1) in. and (2, 3) inches.

Which of the marksmen was the probable winner?"

SOLUTION BY R. J. ADCOCK, ROSEVILLE, ILL.

Let x and y be the coordinates of the bull's eye with reference to the arbitrary origin, and (x_1, y_1), (x_2, y_2), (x_3, y_3) those of A' shots. Then its most probable position is that which makes

$$x^2+y^2+(x-1)^2+(y-1)^2+(x-2)^2+(y-3)^2+(x-x_1)^2+(y-y_1)^2$$
$$+(x-x_2)^2+(y-y_2)^2+(x-x_3)^2+(y-y_3)^2 = \text{a min.},$$

that is, $x^2+y^2+(x-1)^2+(y-1)^2+(x-2)^2+(y-3)^2+1+4+9=$ a min. $\therefore x=1$, $y=\frac{4}{3}$, and the sum of B's distances is 3.96 in., and as the sum of A's distances is 6, $\therefore B$ is the probable winner.

[Mr. Kummell's solution of this problem is too extended for our space in this No. but will appear in a future No.]

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316. "Required the value of

$$\frac{\log x\sqrt{(1-a^2x^2)}}{\sqrt{(1-x^2)}},\text{ where } x=1,\text{ and } a=\log(1-x)."$$

SOLUTION BY EASTWOOD.

Put $\sqrt{(1-a^2x^2)}=\sqrt{(1+ax)}\sqrt{(1-ax)}=u$, and $\sqrt{(1-x^2)}=v$. We shall find

$$\frac{du}{dx}=\frac{\sqrt{(1-ax)}}{2\sqrt{(1+ax)}}\log x\left[\log a-\frac{x}{1-x}\right]+\frac{\sqrt{(1+ax)}}{2\sqrt{(1-ax)}}\log x\left[\frac{x}{1-x}-\log a\right]$$
$$+\frac{\sqrt{(1+ax)}\sqrt{(1-ax)}}{x},$$

$$\frac{dv}{dx}=-\frac{x}{\sqrt{(1-x^2)}}.$$

When $x=1$, $u=0$, $v=0$, $du\div dx=0$, since $\log x=0$, $dv\div dx=\infty$.

Hence $\left[\frac{u}{v}\right]=\frac{0}{0}$, $\left[\frac{du}{dv}\right]=\frac{0}{\infty}=0$, and so on for higher diff. coefficients.

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317. "If the scale of relation of a recurring series be $a_n-7a_{n-1}+12a_{n-2}=0$, and if $u_0=2$, $u_1=7$, find u_n, and the sum of $u_0+u_1+\dots+u_{n-1}$."

SOLUTION BY WILLIAM HOOVER, WAPAKONETA, O.

The given scale of relation is an equation in Finite Differences. Integrating and denoting by u_n the general term,

$$u_n=C_1(4)^n+C_2(3)^n. \tag{1}$$

When $n=0$, $u_n=u_0=2$, and when $n=1$, $u_n=u_1=7$, hence by substitution in (1) we have

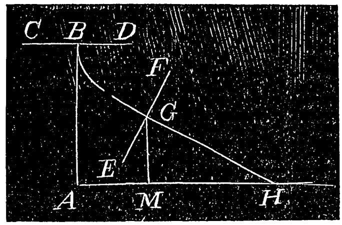
$$C_1+C_2=2, \tag{2}$$
$$4C_1+3C_2=7, \tag{3}$$

whence $C_1=1$, and $C_2=1$. $\therefore u_n=4^n+3^n$. We have $u_{n-1}=(4)^{n-1}+(3)^{n-1}$. $\therefore \Sigma u_{n-1}=\Sigma(4)^n+\Sigma(3)^n=\frac{1}{3}(4)^n+\frac{1}{2}(3)^n+C$.

When $n=1$, $u_0=2$; $\therefore C=-\frac{5}{6}$. $\therefore u_0+u_1+\dots+u_{n-1}=\frac{1}{3}(4)+\frac{1}{2}(3)^n-\frac{5}{6}$, the sum required.

318. "A man takes hold of the end of a cart tongue and travels off at right angles to the direction in which the tongue originally lay. Required the equation of the curve made by the middle of the axle."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MISSOURI.



Let ACD represent the original position of the cart, AB being the tongue, and CD the axle, and let HEF represent any other position of the cart. Draw GM perpendicular to AH. Let $HG = AB = a$, $AM = x$, $GM = y$. Then we have $MH = \sqrt{(a^2 - y^2)}$, and $GM = HM \tan GHM$, or $y = -\sqrt{(a^2 - y^2)} \times dy \div dx$, or $dx = -\sqrt{(a^2 - y^2)} dy \div y$ Integrating, and observing that when $y = a$, $x = 0$, we have

$$x = a \log\left(\frac{a - \sqrt{(a^2 - y^2)}}{y}\right) - \sqrt{(a^2 - y^2)}, \text{ the req'd eq'n.}$$

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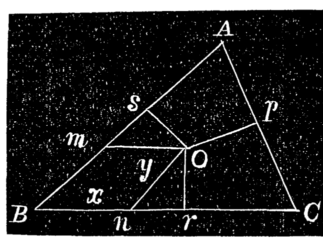
319. "Within a triangle to determine a point so that if the three pependiculars are let fall from it upon the sides of the triangle, the latter will be divided into three equal parts."

SOLUTION BY PROF. CASEY.



Let ABC be the $\triangle$; take BC, BA as axes of x, y. Let O be the required point; draw On, Om, parallel to the axes, and let $Bn = x$ and $On = y$. The area $BsOr$ is given. $y \times \sin B = Or$ and $y \cos B = nr$; $\therefore$ $x + y \cos B = Br$, $x \sin B = Os$ and $y + x \cos B = Bs$. Whence $(x + y \cos B) y \sin B + (y + x \cos B) \times x \sin B = 2^{ce} BsOr = \frac{4}{9} \triangle BAC$, or $x^2 + y^2 + 2xy \sec B = \frac{4}{9} \triangle \div \sin B \cos B$ = a given quantity, and $\therefore$ the locus of O is a conic, and for a like reason in regarding the area $ApOs$, the locus of O is also a conic, which determines the point O.

PROBLEMS.

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320. *By Octavian L. Mathiot, Baltimore, Md.*—The transverse and conjugate axes of an ellipse being given, to find the diameter of the circular base, and the altitude, of the right cone, and where to pass a plane so as to produce the given ellipse.